# Working with Conjunctions

From ‘P & Q’ both ‘P’ and ‘Q’ follow. We will introduce a rule that enables us to move from any conjunction p & q to either conjunct. We will call this rule &Elimination or, for short, &E.

&E

| | | |
|---|---|---|
| From | p & q | p & q |
| To | p | q |

Here is the way a simple derivation using this rule would look:

| | |
|---|---|
| 1. P & Q | Premise |
| 2. P | 1 &E |

Here we have shown that P & Q |- P, that is, that ‘P’ follows from ‘P & Q’. Note that the rule applies to any conjunction regardless of the complexity of the conjuncts:

| | |
|---|---|
| 1. (P & R) & Q | Premise |
| 2. P & R | 1 &E |

There is (at this point at any rate) basically only one kind of mistake you can make. This rule only applies to whole lines which are conjunctions. Consider the following:

| | |
|---|---|
| 1. (P & R) & Q | Premise |
| 2. R | 1 &E - ERRONEOUS |

This is erroneous, since ‘R’ is not either the left or the right conjunct of the line to which you appeal. Here we could, in fact, get to ‘R’ by doing the following:

| | |
|---|---|
| 1. (P & R) & Q | Premise |
| 2. P & R | 1 &E |
| 3. R | 2 &E |

It is indeed obvious that 'R' does follow from this premise, but when we use a deductive system we do not appeal to our intuitions. We ensure that our derivations are correct by applying our rules. Consider the following:

| | |
|---|---|
| 1. (P & R) v Q | Premise |
| 2. P & R | 1 &E - ERRONEOUS |

Again our &E applies only to whole lines that are conjunctions. The problem here is that the line appealed to is not a conjunction; it is instead a disjunction. This error is in a sense much worse than the preceding one. 'P & R' does not follow from this premise.

As well as eliminating conjunctions we can introduce them. We will call the rule that we use to introduce them &Introduction or, for short, &I.

&I

| | | |
|---|---|---|
| From | p, q | p, q |
| To | p & q | q & p |

Notice that while the previous rule appealed to only one previous line, this rule appeals to two previous lines. Consider the following:

| | |
|---|---|
| 1. P | Premise |
| 2. Q | Premise |
| 3. P & Q | 1, 2 &I |

Clearly if 'P' is true and 'Q' is true then 'P & Q' is true. I said that this rule involves an appeal to two previous lines. But in fact, strange though it seems, the "two" lines appealed to may be one and the same. Let us see why. It is clear that where p is true p & p is true. Here is a case:

| | |
|---|---|
| 1. Q | Premise |
| 2. Q & Q | 1, 1 &I |

Notice that in our justification we still mention two lines, but these lines are one and the same. From p it does indeed follow that p & p. If our system is to be complete we need to have some way of showing this.As just seen we do. But there is a general point in the background here. A binary connective * is **idempotent** if and only if p * q is logically equivalent to q * p. '&' is idempotent.

Here are some derivations we can do now that we have both rules. It is, for example, obvious that p & q is true if and only if q & p is true. That is, p & q and q & p are logically equivalent. A binary connective * is **commutative** if and only if p * q is logically equivalent to q * p. So the binary connective '&' is, here continuing to use the language with which you may be familiar from mathematics, commutative. Here is one example of how we would in take advantage of this in our deductive system:

| | |
|---|---|
| 1. P & R | Premise |
| 2. P | 1 &E |
| 3. R | 1 &E |
| 4. R & P | 2, 3 &I |

Here we have shown that P & R I- R & P.

A binary connective * is **associative** if and only if p * (q * r) is logically equivalent to (p * q) * r. Since p & (q & r) is logically equivalent to (p & q) & r), '&' is associative. It is quite simple in our system to show, for example, that (A <-> B) & (C & D) I- ((A <-> B) & C) & D.

| | |
|---|---|
| 1. (A <-> B) & (C & D) | Premise |
| 2. A <-> B | 1 &E |
| 3. C & D | 1 &E |
| 4. C | 3 &E |
| 5. D | 3 &E |
| 6. (A <-> B) & C | 2,4 &I |
| 7. ((A <-> B) & C) & D | 5,6 &I |

Notice that &E does apply to 1. It applies to any line that is a conjunction. It does not matter how complex the conjuncts are.