Working with Conditionals

Our if … then sentences p -> q, which we call conditionals, count as true in every case except the case where p is true and q is false. The sentence p (whatever sentence is on the left) is called the antecedent of the conditional; the sentence q (whatever sentence is on the right) is called the consequent of the conditional. Given p and p -> q it follows that q. We will introduce a rule that enables us to move from any conditional p -> q and p to q. We will call this rule ->Elimination or, for short, ->E. This rule is often called modus ponens, and is sometimes called affirming the antecedent:

->E

From	p -> q
	p
To	q

Here is the way a simple derivation using this rule would look:

1. (A v B) -> Q	Premise
2. A v B	Premise
3. Q	1, 2 ->E

Here we have shown that (A v B) -> Q, A v B |- Q, that is, that ‘Q’ follows from ‘(A v B) -> Q’ and ‘A v B’. Note that the rule applies to any case in which we have a conditional and the antecedent of the conditional regardless of their complexity:

1. ((P & ~R) & Q) -> (~A <-> B)	Premise
2. (P & ~R) & Q	Premise
3. ~A <-> B	1, 2 ->E

Note that we must mention both of the lines appealed to in the justification. As we have seen in the preceding section, it is of the utmost importance to apply the rules only to whole lines. Consider the following:

1. (P & R) & Q	Premise
2. Q -> S	Premise
3. S	1, 2 ->E - ERRONEOUS

‘S’ does indeed follow but not in a single step. Line 1 is not, as it stands, the antecedent of the conditional appealed to. We can, with some additional steps, obtain S:

1. (P & R) & Q	Premise
2. Q -> S	Premise
3. Q	1 &E
4. S	2, 3 ->E

When we use a deductive system we do not appeal to our intuitions. We ensure that our derivations are correct by applying our rules. Next consider the following:

1. (P -> R) <-> Q	Premise
2. P	Premise
3. R	1, 2 ->E - ERRONEOUS

Line one is not a conditional and our ->E rule cannot be applied to it. R does not follow from these premises.

We will eventually introduce a rule, or rather a mode of proof, that we will call ->Introduction. But, because this rule will involve more advanced considerations, we shall postpone its introduction. For now we will introduce a rule that we will call modus tollens, or mt for short. This rule will enable us to move from p -> q and ~q to ~p. This rule is sometimes called denying the consequent.

mt

From	p -> q
	~q
To	~p

Notice that as was the case with the preceding rule, this rule appeals to two previous lines. Consider the following:

1. P -> (R v S)	Premise
2. ~(R v S)	Premise
3. ~P	1, 2 mt

To apply this rule you must have a line that is a conditional and a line that is the negation of the consequent of that conditional. The only line that you can obtain by applying this rule is a line that is the negation of the antecedent of the conditional. Consider the following supposed derivation:

1. P -> ~R	Premise
2. R	Premise
3. ~P	1, 2 mt - ERRONEOUS

The rule mt does not apply here since line 2 is not the negation of the consequent of the conditional that is line 1. As it happens, '~P' does follow from lines 1 and 2 but we cannot show this in our deductive system with the rules that we currently have available. (In a later section we will introduce a rule that will allow us to move from p to ~~p. Once we have that rule we will, with an additional step, be able to derive '~P'.)

A related point applies to the following supposed derivation:

1. ~P -> R	Premise
2. ~R	Premise
3. P	1, 2 mt - ERRONEOUS

What we could get via an application of mt is '~~P'. Recall that mt allows us only to obtain a line that is the negation of the antecedent of the antecedent. (In a later section we will introduce a rule that allows us to move from ~~p to p. Once we have that rule, we will, with an additional step, be able to derive 'P'.)

Here are some of the derivations that we can do now that we have four rules. We will show that P & ~R, P -> S, Q -> R |- S & ~Q.

1. P & ~R	Premise
2. P -> S	Premise
3. Q -> R	Premise
4. P	1 &E
5. ~R	1 &E
6. ~Q	3, 5 mt
7. S	2, 4 ->E
8. S & ~Q	6, 7 &I

This derivation is not unique. No derivation is. Typically there is no one particular order in which the rules must be applied. Here is a derivation that differs slightly from the preceding one.

1. P & ~R	Premise
2. P -> S	Premise
3. Q -> R	Premise
4. P	1 &E
5. S	2, 4 ->E
6. ~R	1 &E
7. ~Q	3, 6 mt
8. S & ~Q	5, 7 &I

If you cannot initially see how to construct a derivation of a conclusion that you wish to derive, it is sometimes helpful to apply ->E, mt or &E if you can. Sometimes this will lead to derivations that have lines that are not later used. But that does not matter as long as the rules have been correctly applied. Here is a trivial example in which you wish to show that (P & ~R) & Q, P -> S |- S :

1. (P & ~R) & Q	Premise
2. P -> S	Premise
3. Q	1 &E
4. P & ~R	1 &E
5. ~R	4 &E
6. P	4 &E
7. S	2, 6 ->E

Note that in reaching the conclusion you do not use either line 3 or line 5. So they need not have been included in order to complete the derivation. But that does not mean that the derivation is defective. It is a perfectly legitimate derivation, albeit a somewhat inelegant one.

Here is another sample derivation where we show that ~(P v (R & S)), (Q & C) -> (P v (R & S)), A -> (Q & C) |- ~A & ~(Q & C):

1. ~(P v (R v S))	Premise
2. (Q & C) -> (P v (R & S))	Premise
3. A -> (Q & C)	Premise
4. ~(Q & C)	1, 2 mt
5. ~A	3, 4 mt
6. ~A & ~(Q & C)	4, 5 &I

Note that the following derivation is defective:

1. ~P v (R v S)	Premise
2. (Q & C) -> (P v (R & S))	Premise
3. A -> (Q & C)	Premise
4. ~(Q & C)	1, 2 mt - ERRONEOUS
5. ~A	3, 4 mt
6. ~A & ~(Q & C)	4, 5 &I

The problem here is that line 1 is not the negation of the consequent of line 2. Indeed line 1 is not a negation at all. It is a disjunction in which the left disjunct is '~P'. When using a deductive system you must pay careful attention to the form or structure of the lines.

Consider the following two supposed derivations:

1. P -> Q	Premise
2. Q	Premise
3. P	1, 2 ->E - ERRONEOUS

1. P -> Q	Premise
2. ~P	Premise
3. ~Q	1, 2 mt - ERRONEOUS

In the first example we are given a conditional and a line that is the consequent of that conditional. This move, often called **affirming the consequent**, is illegitimate. Remember that to use ->E you must have a line that is a conditional and another line that is the antecedent of the conditional. In the second example we are given a conditional and a line that is the negation of the antecedent of that conditional. This move, often called **denying the antecedent**, is also illegitimate. To use mt you must have a line that is a conditional and another line that is the negation of the consequent of that conditional.