Working with Disjunctions

Our 'or' sentences p v q, which we call disjunctions, count as true in every case except the case where both p and q are false. The sentence p (whatever sentence is on the left) is called the left disjunct, and the one on the right is called the right disjunct. Given ~p and p v q it follows that q. Similarly, given ~q and p v q it follows that p. We will introduce a rule that enables us make these moves, and call it disjunctive syllogism (ds for short):

DS

From	p v q	p v q
	~p	~q
To	q	p

Here is the way a simple derivation using this rule would look:

1. (A & B) v Q	Premise
2. ~Q	Premise
3. A & B	1, 2 ->E

Here we have shown that (A & B) v Q, ~Q |- A & B. Note that this rule applies only in cases in which we have a disjunction and a line that is the negation of either the left or the right disjunct. Given that, we can add the other disjunct as a line. Here is a slightly longer derivation that uses our new rule and some of the previous ones.

1. P & ~R	Premise
2. S v R	Premise
3. S -> A	Premise
4. ~R	1 &E
5. S	2, 4 DS
6. A	3, 5 ->E
7. P	1 &E
8. P & S	5, 7 &I

Consider the following supposed derivation:

1. ~P v Q	Premise
2. P	Premise
3. Q	1, 2 DS - ERRONEOUS

Line 2 is not the negation of a disjunct in line 1. 'Q' does indeed follow, but not with the rules we currently have available. (The rule we need is ~~I. We will introduce it later.) As per usual make sure that you apply the rules only to whole lines.

When we use a deductive system we do not appeal to our intuitions. We ensure that our derivations are correct by applying our rules. Next consider the following:

1. (P v R) & Q	Premise
2. ~P	Premise
3. R	1, 2 DS - ERRONEOUS

Line one is not a disjunction, and DS cannot be applied to it. In this case R does follow from these premises. Simply use &E on line 1 to obtain P v R as a line, and then use DS.

Here is one more derivation:

1. (P & ~Q) v R	Premise
2. ~R	Premise
3. P -> A	Premise
4. Q v S	Premise
5. P & ~Q	1, 2 DS
6. P	5 &E
7. ~Q	5 &E
8. S	4, 7 DS
9. A	3, 6 ->E
10. S & A	8, 9 &I

Note that in this case, with the rules we have, there is nothing you can do to start your derivation save applying DS. Again, if you get stuck you should apply any rule that you can. In this case once you have applied DS you should see that various paths lead to the conclusion. Here is another equally good derivation.

1. (P & ~Q) v R	Premise
2. ~R	Premise
3. P -> A	Premise
4. Q v S	Premise
5. P & ~Q	1, 2 DS
6. P	5 &E
7. A	3, 6 ->E
8. ~Q	5 &E
9. S	4, 8 DS
10. S & A	7, 9 &I

You should be able to see that our ‘v’ is such that where p is true p v q is true, and where p is true q v p is true. q can be any sentence you wish. Our system will now incorporate a rule vIntroduction (vI) that enables us to make such a move. This rule is also sometimes called addition.

vI

From	p	p
To	p v q	q v p

Here is an example of a case where this rule would be very useful:

1. (P v R) -> Q	Premise
2. P	Premise
3. P v R	2 vI
4. Q	1, 3 ->E

We could not have shown that (P v R) -> Q, P |- Q without this new rule.

But how do you know when to use this rule and what to add? After all, the rule can be applied again and again, adding as disjuncts any sentence whatsoever.There is no simple answer. Generally you should only use this rule when, as in the above example, you see how it would help you to proceed in your derivation toward whatever it is that you are attempting to derive. A sign, not conclusive but fairly reliable, that you will have to use vI is that your conclusion contains some basic sentence that does not occur in any of the premises. Suppose, for example, that you are asked to show that P, P -> Q |- Q v A. Here you would simply apply ->E and then use a vI. You would add A simply because that leads you to the conclusion that you want to reach. Another sign, again not conclusive, that you may have to use vI is that you find that you need a disjunction to proceed in your derivation. You know that if you can obtain one of the disjuncts you can always obtain the disjunction you need by using vI. Here is another example of a derivation where we use vI:

1. (P & ~Q) v R	Premise
2. ~R	Premise
3. (P v S) -> A	Premise
4. (A v C) -> D	Premise
5. P & ~Q	1, 2 DS
6. P	5 &E
7. P v S	6 vI
8. A	3, 7 ->E
9. A v C	8 vI
10. D	4, 9 ->E
11. (F <-> G) v D	10 vI

Here we have shown that (P & ~Q) v R, ~R, (P v S) -> A, (A v C) -> D |- (F <-> G) v D. This also illustrates the point that you can add as complex a disjunct as you need.

Now that we have this rule, we can show that if you have available lines p and ~p you can obtain any sentence q that you want. (This is why it is sometimes said that anything follows from a contradiction.)

Here is a sketch of this trick (k, m, n indicate line numbers):

k	p	
m	~p	
n	p v q	k vI
n+1	q	m, n DS

Here is one example where we show that A v B, ~A & ~B I- C & D:

1. A v B	Premise
2. ~A & ~B	Premise
3. ~A	2 &E
4. ~B	2 &E
5. A	1, 4 DS
6. A v (C & D)	5 vI
7. C & D	3, 6 DS

We will find that there are many occasions on which this trick is very useful.

We have one more rule that concerns disjunctions to introduce into our current repertoire. This will be our vElimination (vE) rule. It is a variant of a rule often called constructive dilemma. Here it is:

vE

From	p v q
	p -> r
	q -> r
To	r

Note that this is the first (and will be the only) rule we have that appeals to three preceding lines. Suppose we wish to show that (P v Q) & R, P -> S, Q -> S, (R & S) -> A I- A. Here is a derivation:

1. (P v Q) & R	Premise
2. P -> S	Premise
3. Q -> S	Premise
4. (R & S) -> A	Premise
5. P v Q	1 &E
6. R	1 &E
7. S	2, 3, 5 vE
8. R & S	6, 7 &I
9. A	4, 8 ->E

Here is another derivation in which we show that R -> (D -> S), (A & B) -> (D -> S), (R v (A & B)) & (Q & D) I- (S & Q) v C:

1. R -> (D -> S)	Premise
2. (A & B) -> (D -> S)	Premise
3. (R v (A & B)) & (Q & D)	Premise
4. R v (A & B)	3 &E
5. Q & D	3 &E
6. D -> S	1, 2, 4 vE
7. D	5 &E
8. Q	5 &E
9. S	6, 7 ->E
10. S & Q	8, 9 &I
11. (S & Q) v C	10 vI

We noted, when we discussed conjunctions, that '&' is commutative, idempotent and associative. Let us consider 'v'. Note that p v q is indeed logically equivalent to q v p. But at this point we do not have the resources to establish this. As well, p v p is logically equivalent to p. And p v (q v r) is logically equivalent to (p v q) v r. We also currently lack the resources to establish either of these.

This completes our discussion of rules that pertain specifically to disjunctions.