Working with Double Negations

We have not as yet paid much attention to double negations, although we noted that we needed some rules to enable us to work with them. You should know that p and ~~p are logically equivalent. Here is the first, our elimination rule:

~~E

From	~~p
To	p

As long as you apply this rule only to a whole line that is a double negation, you should have no problem. Here is a fairly simple derivation using this rule:

1. P v ~~~Q	Premise
2. ~~~P & (R -> Q)	Premise
3. ~~~P	2 &E
4. R -> Q	2 &E
5. ~P	3 ~~E
6. ~~~Q	1, 5 DS
7. ~Q	6 ~~E
8. ~R	4, 7 MT

Here is a supposed derivation:

1. ~(~P & R)	Premise
2. P & R	1 ~~E - ERRONEOUS

Line 1 is not a double negation -- it is instead the negation of the conjunction ~P & R. Line 2 does not in fact follow from line 1 at all, so there is no way to “repair” this supposed derivation. Consider the following:

1. ~~P & R	Premise
2. P & R	1 ~~E - ERRONEOUS

Though a subsentence of line 1 is a double negation line 1 is itself a conjunction rather than a double negation. Here you can "repair" the derivation:

1. ~~P & R	Premise
2. ~~P	1 &E
3. R	1 &E
4. P	2 ~~E
5. P & R	3, 4 &I

Our ~~I rule is exactly what you would expect:

~~I

From	p
To	~~p

Here is a derivation in which we use this rule:

1. ~~P v Q	Premise
2. ~P & (R -> ~Q)	Premise
3. ~P	2 &E
4. R -> ~Q	2 &E
5. ~~~P	3 ~~I
6. Q	1, 5 DS
7. ~~Q	6 ~~I
8. ~R	4, 7 MT

Both of these rules are very easy, so we shall provide only one more sample derivation:

1. ~~P <-> ~Q	Premise
2. (P -> R) & ~~~Q	Premise
3. (~~P -> ~Q) & (~Q -> ~~P)	1 <->E
4. P -> R	2 &E
5. ~~~Q	2 &E
6. ~Q -> ~~P	3 &E
7. ~Q	5 ~~E
8. ~~P	6, 7 ->E
9. P	8 ~~E

Here we have shown that ~~P <-> ~Q, (P -> R) & ~~~Q |- P. Note that line 4 is not later used. However that does not mean that the derivation is defective, it is simply not the shortest one possible.

This completes our introduction to the basic rules of the system. We will next introduce two new rules, or modes of proof as we prefer to say, that are conceptually different.