

Working with Theorems and Logical Equivalences

You already know that certain sentences, for example $P \vee \sim P$ and $P \rightarrow P$, are tautologies or logical truths. They come out true under all assignments of truth values. Our deductive system allows us to establish that certain sentences are tautologies by way of establishing that they are theorems of our system. But what is a theorem and how do we establish one? There is nothing in our statement of the modes of proof that allow us to introduce assumptions that require us to have any premises. An assumption can be introduced at any point in a derivation. So when we are establishing a theorem, our initial line will be indented one column. If we can terminate and discharge that assumption the final line will be in the main column. Here is a quite trivial example:

| | | |
|----|-------------------|---------------------|
| 1. | P | * A |
| 2. | P | * 1 Reit |
| 3. | $P \rightarrow P$ | 1-2 \rightarrow I |

Here we have established our first theorem: we have shown that $\vdash P \rightarrow P$. Note that there are no premises. In the common jargon, we will say that we have derived it from the null or empty set of premises. Do not worry about this jargon — from your standpoint it simply means that there are no premises. Since our system is a complete one where p is any tautology, there exists a derivation of p from the empty set of premises. That is to say, where p is any tautology, p is a theorem of our system. Since our system is a sound one, it is also true that where p is any theorem p is a tautology. Though we will not prove it here, our system would be unsound were there any theorems that were not tautologies.

Let us now learn how to establish theorems, holding off on a discussion of biconditionals until later. Note that no basic or atomic sentence is a tautology, hence no basic or atomic sentence is a theorem. So all theorems will be conjunctions, conditionals, disjunctions, or negations. Note that a conjunction $p \ \& \ q$ will be a theorem if and only if both p and q are theorems. This follows, given the properties

of our system, from the fact that $p \ \& \ q$ is a tautology if and only if both p and q are tautologies. So to show that $\vdash p \ \& \ q$, construct a derivation in which you obtain p and obtain q in the main column and then use $\&I$. To establish that some conditional $p \rightarrow q$ is a theorem it is generally expedient to set up a $\rightarrow I$. That is, make p your initial assumption and try to derive q . Then you will have $p \rightarrow q$ in the main column. To show that a disjunction $p \vee q$ is a theorem, it is generally expedient to set up a $\sim E$: that is, assume $\sim(p \vee q)$. To show that a negation $\sim p$ is a theorem, set up a $\sim I$: that is, assume p .

Let us now look at some examples of derivations. We will not attempt to obtain the shortest derivation. Instead we will utilize the various suggestions we have made. As you gain experience you will find that you may see shorter ways to establish some particular theorem. Let us first show that $\vdash P \rightarrow (Q \rightarrow P)$.

| | | |
|----|---|---|
| 1. | P | A |
| 2. | Q | A |

Here we made line 2 the assumption of Q since the sentence we need is $Q \rightarrow P$, a sentence that is itself a conditional. The rest of the derivation is straightforward:

| | | |
|----|-----------------------------------|-----------------------|
| 1. | P | * A |
| 2. | Q | * A |
| 3. | P | * 1 Reit |
| 4. | $Q \rightarrow P$ | * 2-3 $\rightarrow I$ |
| 5. | $P \rightarrow (Q \rightarrow P)$ | 1-4 $\rightarrow I$ |

$P \vee \sim P$ is a frequently used example of a tautology. Let us show that $\vdash P \vee \sim P$. We will use a $\sim E$.

| | | |
|----|-----------------------|---|
| 1. | $\sim(P \vee \sim P)$ | A |
|----|-----------------------|---|

You might find yourself stumped at this point. What can you do next? You need a contradiction to complete the derivation. In this case, P is the only letter around, so try to obtain $P \ \& \ \sim P$. Since this is a conjunction you can get it by establishing them separately and then using $\&I$.

| | | |
|-----|--|----------------|
| 1. | $\sim(P \vee \sim P)$ | * A |
| 2. | P | * A |
| 3. | $P \vee \sim P$ | * 2 vI |
| 4. | $(P \vee \sim P) \& \sim(P \vee \sim P)$ | * 1,3 &I |
| 5. | $\sim P$ | * 2-4 \sim I |
| 6. | $\sim P$ | * A |
| 7. | $P \vee \sim P$ | * 6 vI |
| 8. | $(P \vee \sim P) \& \sim(P \vee \sim P)$ | * 1, 7 &I |
| 9. | P | * 6-8 \sim E |
| 10. | $P \& \sim P$ | * 5, 9 &I |
| 11. | $P \vee \sim P$ | 1-10 \sim E |

In a preceding section we showed that $\sim(P \& Q) \text{ I- } \sim P \vee \sim Q$. Here is that derivation:

| | | |
|-----|--|----------------|
| 1. | $\sim(P \& Q)$ | Premise |
| 2. | $\sim(\sim P \vee \sim Q)$ | * A |
| 3. | $\sim P$ | * A |
| 4. | $\sim P \vee \sim Q$ | * 3 vI |
| 5. | $\sim(\sim P \vee \sim Q) \& (\sim P \vee \sim Q)$ | * 2, 4 &I |
| 6. | P | * 3-5 \sim E |
| 7. | $\sim Q$ | * A |
| 8. | $\sim P \vee \sim Q$ | * 7 vI |
| 9. | $\sim(\sim P \vee \sim Q) \& (\sim P \vee \sim Q)$ | * 2, 8 &I |
| 10. | Q | * 7-9 \sim E |
| 11. | $P \& Q$ | * 6, 10 &I |
| 12. | $(P \& Q) \& \sim(P \& Q)$ | * 1, 11 &I |
| 13. | $\sim P \vee \sim Q$ | 2-12 \sim E |

It is a characteristic of our system that where $p \text{ I- } q$, $\text{I- } p \rightarrow q$. Where q follows from p , $p \rightarrow q$ will be a tautology. For if q follows from p , then q will be true in any case p is true. So where p is true $p \rightarrow q$ will be true. And where p is false $p \rightarrow q$ will be true. So in any case $p \rightarrow q$ will be true: that is, it is a tautology. It is quite a simple matter to convert the above derivation of q from p to a derivation that shows that $\text{I- } p \rightarrow q$. All that one does is change the status of the initial line from a premise to an assumption. This will be the new derivation:

| | | |
|-----|--|----------------------|
| 1. | $\sim(P \ \& \ Q)$ | * A |
| 2. | $\sim(\sim P \vee \sim Q)$ | * A |
| 3. | $\sim P$ | * A |
| 4. | $\sim P \vee \sim Q$ | * 3 vI |
| 5. | $\sim(\sim P \vee \sim Q) \ \& \ (\sim P \vee \sim Q)$ | * 2, 4 &I |
| 6. | P | * 3-5 \sim E |
| 7. | $\sim Q$ | * A |
| 8. | $\sim P \vee \sim Q$ | * 7 vI |
| 9. | $\sim(\sim P \vee \sim Q) \ \& \ (\sim P \vee \sim Q)$ | * 2, 8 &I |
| 10. | Q | * 7-9 \sim E |
| 11. | P & Q | * 6, 10 &I |
| 12. | (P & Q) & $\sim(P \ \& \ Q)$ | * 1, 11 &I |
| 13. | $\sim P \vee \sim Q$ | * 2-12 \sim E |
| 14. | $\sim(P \ \& \ Q) \ \rightarrow (\sim P \vee \sim Q)$ | 1-13 \rightarrow I |

Note that this tactic is perfectly general. Any time you can show $p \vdash q$, you can show $\vdash p \rightarrow q$ and vice versa.

Let's do one more derivation before turning to biconditionals. We will show that $\vdash \sim(P \ \& \ \sim Q) \rightarrow (P \rightarrow Q)$:

| | | |
|----|---|-----------------------|
| 1. | $\sim(P \ \& \ \sim Q)$ | * A |
| 2. | P | * A |
| 3. | $\sim Q$ | * A |
| 4. | P & $\sim Q$ | * 2, 3 &I |
| 5. | (P & $\sim Q$) & $\sim(P \ \& \ \sim Q)$ | * 1, 4 &I |
| 6. | Q | * 3-5 \sim E |
| 7. | P \rightarrow Q | * 2-6 \rightarrow I |
| 8. | $\sim(P \ \& \ \sim Q) \rightarrow (P \rightarrow Q)$ | 1-7 \rightarrow I |

Certain sentences p and q are logically equivalent. Under any assignment of truth values, either both are true or both are false. Note that where p and q are logically equivalent $p \leftrightarrow q$ is a tautology, since p, q will always either both be true or both be false. So we can in effect show, using our system, that p, q are logically equivalent by showing that the biconditional $p \leftrightarrow q$ is a theorem. Establishing

biconditionals involves no new principles. As we noted, we typically establish a biconditional $p \leftrightarrow q$ by showing that $p \rightarrow q$ and showing that $q \rightarrow p$. So to establish that $\vdash p \leftrightarrow q$, establish both of $\vdash p \rightarrow q$ and $\vdash q \rightarrow p$, then use $\&I$ and $\leftrightarrow I$. In the previous example we showed that $\vdash \sim(P \& \sim Q) \rightarrow (P \rightarrow Q)$. As it happens, $\sim(P \& \sim Q)$ and $P \rightarrow Q$ are logically equivalent, so we should be able to show that $\vdash \sim(P \& \sim Q) \leftrightarrow (P \rightarrow Q)$. All that we will have to do is continue our previous derivation:

| | | |
|-----|---|-------------------------|
| 1. | $\sim(P \& \sim Q)$ | * A |
| 2. | P | * A |
| 3. | $\sim Q$ | * A |
| 4. | $P \& \sim Q$ | * 2, 3 &I |
| 5. | $(P \& \sim Q) \& \sim(P \& \sim Q)$ | * 1, 4 &I |
| 6. | Q | * 3-5 $\sim E$ |
| 7. | $P \rightarrow Q$ | * 2-6 $\rightarrow I$ |
| 8. | $\sim(P \& \sim Q) \rightarrow (P \rightarrow Q)$ | 1-7 $\rightarrow I$ |
| 9. | $P \rightarrow Q$ | * A |
| 10. | $P \& \sim Q$ | * A |
| 11. | P | * 10 &E |
| 12. | $\sim Q$ | * 12 &E |
| 13. | Q | * 9, 11 $\rightarrow E$ |
| 14. | $Q \& \sim Q$ | * 12, 13 &I |
| 15. | $\sim(P \& \sim Q)$ | * 10-14 $\sim I$ |
| 16. | $(P \rightarrow Q) \rightarrow \sim(P \& \sim Q)$ | 9-15 $\rightarrow I$ |
| 17. | $((P \rightarrow Q) \rightarrow \sim(P \& \sim Q)) \&$ $(\sim(P \& \sim Q) \rightarrow (P \rightarrow Q))$ | 8, 16 &I |
| 18. | $\sim(P \& \sim Q) \leftrightarrow (P \rightarrow Q)$ | 17 $\leftrightarrow I$ |

This is long, but does not require any special tricks.

Let us now show that that $\vdash (\sim P \vee Q) \leftrightarrow (P \rightarrow Q)$:

| | | |
|-----|---|-------------------------|
| 1. | $\sim P \vee Q$ | * A |
| 2. | P | * A |
| 3. | $\sim\sim P$ | * 2 $\sim\sim$ I |
| 4. | Q | * 1, 3 DS |
| 5. | $P \rightarrow Q$ | * 2-4 \rightarrow I |
| 6. | $(\sim P \vee Q) \rightarrow (P \rightarrow Q)$ | 1-5 \rightarrow I |
| 7. | $P \rightarrow Q$ | * A |
| 8. | $\sim(\sim P \vee Q)$ | * A |
| 9. | $\sim P$ | * A |
| 10. | $\sim P \vee Q$ | * 9 \vee I |
| 11. | $(\sim P \vee Q) \& \sim(\sim P \vee Q)$ | * 8, 10 $\&$ I |
| 12. | P | * 9-11 \sim E |
| 13. | Q | * 7, 12 \rightarrow E |
| 14. | $\sim P \vee Q$ | * 13 \vee I |
| 15. | $\sim(\sim P \vee Q) \& (\sim P \vee Q)$ | * 8, 14 $\&$ I |
| 16. | $\sim P \vee Q$ | * 8-15 \sim E |
| 17. | $(P \rightarrow Q) \rightarrow (\sim P \vee Q)$ | 7-16 \rightarrow I |
| 18. | $((P \rightarrow Q) \rightarrow (\sim P \vee Q))$ $\& ((\sim P \vee Q) \rightarrow (P \rightarrow Q))$ | 6, 17 $\&$ I |
| 19. | $(\sim P \vee Q) \leftrightarrow (P \rightarrow Q)$ | 18 \leftrightarrow I |

There are, of course, theorems and logical equivalences that are very difficult to establish. But you should have no particular problem in establishing some of the more common of them.

An individual sentence p may be consistent or inconsistent. p is consistent if and only if there is at least one assignment of truth values under which it is true. Hence an individual sentence p is inconsistent if and only if there is no assignment under which it is true. Note that the negation of an inconsistent sentence is a tautology or logical truth. We have noted already that, since our system is a complete and sound one, all and only tautologies are theorems. We can then establish, by the use of our system, that a given sentence p is inconsistent by way of showing that the negation of the sentence $\sim p$ is a theorem.

But suppose that you are given some very complex sentence p that you did not know to be consistent or know to be inconsistent. You could attempt to derive the negation $\sim p$ of the sentence as a theorem. If you did, you would of course know that the sentence p was inconsistent. But what if you failed to derive $\sim p$? That you failed does not in and of itself tell us whether you overlooked a way of establishing $\sim p$ (given that p was inconsistent) or whether you failed because there is no such derivation, because $\sim p$ is not a theorem. A similar point obtains (it is in effect the same point) with respect to the question of whether a given sentence is a tautology. If it is, then there is a derivation of that sentence. But your failure to find a derivation might mean that you overlooked a way or that it is not a tautology. If you are in doubt, in these cases it is generally better to use truth tables or some other device to determine if the sentence is inconsistent or if the sentence is a tautology.

A set of sentences (we will only concern ourselves with sets that have only a finite number of members) is consistent if and only if there is at least one assignment of truth values in which all the sentences that are members of the set are true. Consequently a set of sentences is inconsistent if and only if there is no assignment of truth values in which all the members of the set are true. There are two ways in which we can show, by using our deductive system, that an inconsistent set of sentences is inconsistent. One way is to put in as premises of an argument each of the members of the set. The order in which you enter the premises does not matter. If the set is inconsistent, there will be a way of deriving a contradiction $p \ \& \ \sim p$ from those premises. That there is such a way is a consequence of the completeness of our system. Any argument with an inconsistent set of premises is valid, regardless of what the conclusion is. Since there is no assignment of truth values in which all the premises are true, there is no assignment of truth values in which all the premises are true while the conclusion is false. Again, though: attempting to derive a contradiction is not a good way to discover whether a set of sentences is inconsistent. Our situation is the same as that in which we considered only a single sentence. A failure to derive a contradiction may

mean either that you have overlooked something or that the set is not inconsistent. There is another way to establish that an inconsistent set of sentences is inconsistent. Typically we will pick out sets by providing a list $p, q, r, s \dots$ of the members of that set. Form a conjunction of the members by appending each member of the set on the right in a conjunction. Where the set has four members as above, the conjunction will be:

$$((p \ \& \ q) \ \& \ r) \ \& \ s$$

(We will call this the standard conjunction — note that this is simply a convention since any such conjunction would do.) Then form the negation of this conjunction:

$$\sim(((p \ \& \ q) \ \& \ r) \ \& \ s)$$

You should at this point be able to see that this sentence, given that the set is inconsistent, must be a tautology. Since our system is a complete one, there is a way of establishing it as a theorem. We can show that an inconsistent set of sentences is inconsistent by showing that the negation of the standard conjunction is a theorem. Here is an example:

$$\text{Set: } P, P \rightarrow Q, Q \rightarrow R, R \rightarrow \sim P$$

$$\text{Standard Conjunction: } ((P \ \& \ (P \rightarrow Q)) \ \& \ (Q \rightarrow R)) \ \& \ (R \rightarrow \sim P)$$

So our task is to show $\vdash \sim(((P \ \& \ (P \rightarrow Q)) \ \& \ (Q \rightarrow R)) \ \& \ (R \rightarrow \sim P))$. We will utilize $\sim I$:

| | | |
|-----|--|------------------------|
| 1. | $((P \ \& \ (P \rightarrow Q))$ $\& \ (Q \rightarrow R)) \ \& \ (R \rightarrow \sim P)$ | * A |
| 2. | $(P \ \& \ (P \rightarrow Q)) \ \& \ (Q \rightarrow R)$ | * 1 &E |
| 3. | $Q \rightarrow R$ | * 2 &E |
| 4. | $P \ \& \ (P \rightarrow Q)$ | * 2 &E |
| 5. | P | * 4 &E |
| 6. | $P \rightarrow Q$ | * 4 &E |
| 7. | $R \rightarrow \sim P$ | * 1 &E |
| 8. | Q | * 5, 6 \rightarrow E |
| 9. | R | * 3, 8 \rightarrow E |
| 10. | $\sim P$ | * 7, 9 \rightarrow E |
| 11. | $P \ \& \ \sim P$ | * 5, 10 &I |
| 12. | $\sim(((P \ \& \ (P \rightarrow Q)) \ \& \ (Q \rightarrow R))$ $\ \& \ (R \rightarrow \sim P))$ | 1-11 \sim I |

Consider the following derivation:

| | | |
|----|------------------------|----------------------|
| 1. | P | Premise |
| 2. | $P \rightarrow Q$ | Premise |
| 3. | $Q \rightarrow R$ | Premise |
| 4. | $R \rightarrow \sim P$ | Premise |
| 5. | Q | 1, 2 \rightarrow E |
| 6. | R | 3, 5 \rightarrow E |
| 7. | $\sim P$ | 4, 6 \rightarrow E |
| 8. | $P \ \& \ \sim P$ | 1, 7 &I |

Note the relation between this derivation, where we show that $P, P \rightarrow Q, Q \rightarrow R, R \rightarrow \sim P \mid P \ \& \ \sim P$, and the preceding one. All we did here was make the premises of the argument the members of the set and then derive a contradiction. As indicated, this is also a way of showing a set to be inconsistent.